

# Stable two-dimensional solitons supported by radially inhomogeneous self-focusing nonlinearity

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We demonstrate that modulation of the local strength of the cubic self-focusing (SF) nonlinearity in the two-dimensional (2D) geometry, in the form of a circle with contrast  $\Delta g$  of the SF coefficient relative to the ambient medium with a weaker nonlinearity, stabilizes a family of fundamental solitons against the critical collapse. The result is obtained in an analytical form, using the variational approximation (VA) and Vakhitov-Kolokolov (VK) stability criterion, and corroborated by numerical computations. For the small contrast, the stability interval of the soliton's norm scales as  $\Delta N \sim \Delta g$  (the replacement of the circle by an annulus leads to a reduction of the stability region by perturbations breaking the axial symmetry). To further illustrate this mechanism, we demonstrate, in an exact form, the stabilization of 1D solitons against the critical collapse under the action of a locally enhanced quintic SF nonlinearity.

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The use of spatially modulated nonlinearities for supporting solitons in optical waveguides and Bose-Einstein condensates (BECs) has drawn much attention in past several years [1, 2]. Various patterns of the local strength of the nonlinearity can be created in photonic crystals [3]-[6], or with the help of inhomogeneous distributions of resonant dopants in optical waveguides [7]. Similar structures can be induced in BEC by means of the Feshbach resonance controlled by nonuniform external fields [8]-[16]. In the latter case, an effective technique for creating necessary field patterns may be provided by arrayed magnetic films [17].

The spatially modulated nonlinearity gives rise to effective *pseudopotentials* [15] which help to create stable spatial solitons [1] in diverse one-dimensional (1D) settings with the cubic [3]-[5], [8]-[15], quadratic [6] and quintic [18] nonlinearities. Stabilizing multidimensional solitons by means of similar techniques is a harder problem. In the 2D geometry, smooth landscapes of the self-focusing (SF) cubic nonlinearity do not allow one to stabilize solitons against the corresponding *critical* [19]-[21] collapse [16]. Nevertheless, the stabilization of 2D fundamental solitons in sufficiently broad parametric regions is possible in circles [22, 23] or annuli [22], as well as in stripe planforms [24], with *sharp edges*. No results for stable 2D vortex solitons, or 3D solitons of any kind, supported by inhomogeneous SF nonlinearities have been reported thus far. On the other hand, the stabilization of all kinds of bright solitons and solitary vortices in any dimension  $D$  is readily provided by the *self-defocusing* cubic nonlinearity, whose local strength is made to grow at large distances ( $r$ ) faster than  $r^D$  [25]. A completely different mechanism for the stabilization of 2D solitons, based on the use of localized gain [26], is possible in dissipative optical media.

The stabilization mechanism for 2D fundamental solitons has been demonstrated in areas carrying the SF cubic nonlinearity and embedded into self-defocusing or linear host media [22]-[24]. A challenging question is whether the stabilization of solitons against the critical collapse may be possible solely due of the contrast between stronger SF in a given area, and a weaker self-attractive nonlinearity in the ambient medium. The objective of this Letter is to demonstrate that such a mechanism works for fundamental solitons trapped in circles and annuli. Thus, we demonstrate the stabilization of the 2D solitons by the *enhanced self-focusing*: while the well-known Townes solitons are unstable against the collapse in the uniform SF medium [21], the creation of a circle or annulus with a larger SF coefficient,  $g$ , gives rise to a family of *stable* 2D solitons. In particular, in the limit of small contrast  $\Delta g$  between the circle or annulus and the host medium, the stability interval for the solitons scales as  $\Delta g$ . To illustrate the genericity of the mechanism, we also report exact results for the stabilization of 1D fundamental solitons against the respective critical collapse induced by the quintic nonlinearity.

The normalized form of the 2D nonlinear Schrödinger (NLS) equation for wave amplitude  $\phi(x, y, z)$  in the medium with the modulated nonlinearity is [22]

$$i\phi_z = -(1/2)\nabla^2\phi - g(r)|\phi|^2\phi, \quad (1)$$

where  $z$  is the propagation distance,  $x$  and  $y$  are the transverse coordinates,  $r = \sqrt{x^2 + y^2}$ , and

$$g(r) = \begin{cases} 1, & \text{at } \rho < r < 2, \\ 1 - \Delta g, & \text{at } r < \rho \text{ and } r > 2, \end{cases} \quad (2)$$

$\rho$  being the inner radius of the annulus (the circle corresponds to  $\rho = 0$ ), while the outer radius is set to be 2 by scaling. Inside the annulus, the SF coefficient is normalized to be 1, while  $1 - \Delta g$  is the background value. In

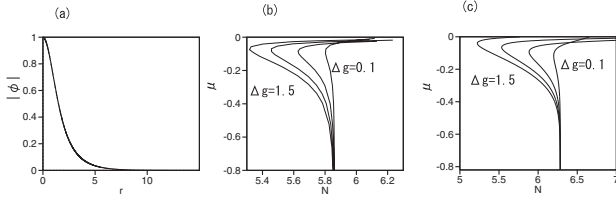


FIG. 1: (a) A stable localized solution for  $\Delta g = 0.5$ ,  $\rho = 0$  and  $\mu = -0.197$ ,  $N = 5.73$ . (b) The propagation constant of numerically found families of the trapped fundamental modes vs. the norm, at fixed values of the nonlinearity contrast between the circle ( $\rho = 0$ ) and surrounding background,  $\Delta g = 1.5, 1.0, 0.5$ , and  $0.1$ . (c) The same as in (b), but as obtained from the VA.

the application to BEC, the evolutionary variable in Eq. (1) is time  $t$ , instead of  $z$ .

Solutions to Eq. (1) in the form of axisymmetric localized modes with propagation constant  $-\mu > 0$  are sought for as  $\phi = u(r) \exp(-i\mu z)$ , with real  $u(r)$  determined by equation

$$\mu u = -(1/2)(u'' + r^{-1}u') - g(r)u^3. \quad (3)$$

It does not make sense to consider  $\Delta g < 0$ , as in that case the annulus is a repulsive structure, that cannot trap stable solitons. On the other hand,  $\Delta g > 1$  implies that the host medium is self-defocusing, which, repelling the wave field, helps to trap and stabilize modes in the SF annulus [22, 23]. In the case of  $\Delta g \gg 1$ , the field is strongly confined to the annulus. Indeed, in a boundary layer adjacent to  $r = 2$ , the solution to Eq. (3) interpolating between the inner one (at  $r < 2$ ), which is characterized by the boundary value of the derivative,  $u'_{r=2}$ , and the outer solution, which quickly decays at  $r > 2$ , is  $u \approx \sqrt{-u'_{r=2}} \left[ (\Delta g)^{1/4} + \sqrt{\Delta g} (-u'_{r=2}) (r - 2) \right]^{-1}$ , with  $u(r = 2) \approx (\Delta g)^{-1/4} \sqrt{-u'_{r=2}}$ . This means that, in the case of  $\Delta g \rightarrow \infty$ , one should look for the inner solution satisfying the boundary condition  $u(r = 2) = 0$ , with arbitrary  $u'_{r=2}$ . The same pertains to the boundary at  $r = \rho$ .

Numerical solutions of Eq. (3) were found by means of the shooting method. Figure 1(a) displays a typical profile of stable fundamental (single-peak) modes. Families of the solutions are presented in Fig. 1(b) by curves for the total power (norm),  $N = 2\pi \int_0^\infty u^2(r) r dr$ , versus  $\mu$ . All the families contain portions that meet the Vakhitov-Kolokolov (VK) stability criterion,  $d\mu/dN < 0$  [20, 21, 27]. Direct simulations of the full 2D equation (1) (without imposing the axial symmetry) confirm that all the solutions satisfying the VK criterion are stable, while those corresponding to  $d\mu/dN > 0$  suffer the collapse or decay, as expected.

A natural tool for the consideration of the present setting is provided by the variational approximation (VA) [22, 28]. To this end, we adopt the Gaussian ansatz,  $u = A \exp\{-r^2/(2W^2)\}$ , with amplitude

$A$ , width  $W$ , and norm  $N = \pi A^2 W^2$ . The substitution of this into the underlying Lagrangian,  $L = 2\pi \int_0^\infty \left[ 2\mu u^2 - (u')^2 + g(r)u^4 \right] r dr$ , yields

$$L = 2\mu N - \frac{N}{W^2} + \frac{N^2}{2\pi W^2} [1 - \Delta g(1 - E_2 + E_1)], \quad (4)$$

where  $E_1 = \exp(-8/W^2)$  and  $E_2 = \exp(-2\rho^2/W^2)$ . Then,  $W$  and  $\mu$  are determined by the variational equations,  $\partial L/\partial W = \partial L/\partial \mu = 0$ . Figure 1(c) shows that the VA produces curves  $\mu(N)$  which are very close to their numerical counterparts. Further, a region in which the VA predicts localized modes that are stable as per the VK criterion (hence they are stable indeed, as confirmed by the aforementioned numerical results) can be found from the variational equation  $\partial L/\partial W = 0$ :

$$(1 + e^{-2}\Delta g)^{-1} < N/(2\pi) < 1 \quad (5)$$

( $N = 2\pi$  is the prediction of the VA for the norm of the Townes soliton [28]). Note that the smallest value of  $N$  in Eq. (5) corresponds to width  $W = 2$ , which coincides with the radius of the circle, the stable branches of the  $\mu(N)$  curves in Fig. 1(c) corresponding to  $W \leq 2$ , i.e., the stable modes are trapped inside the circle. Further, it is seen from Eq. (5) that, for the vanishing contrast between the circle and background,  $\Delta g \rightarrow 0$ , the width of the stability interval shrinks as  $\Delta N \approx 2\pi e^{-2}\Delta g \approx 0.85\Delta g$ . The analysis of numerical solutions yields a very close result for  $\Delta N$ . Thus we conclude that there exists the *stable* subfamily of the 2D solitons in the circle, as long as the SF profile is not completely flat.

Results for the annuli are shown in Fig. 2. The fundamental soliton still features a maximum at  $r = 0$  in Fig. 2(a), in spite of the relatively large radius and depth of the inner hole in that case,  $\rho = 1$  and  $\Delta g = 0.5$ . For a small inner radius  $\rho$ , the VA yields solutions in the range of  $(1 + e^{-2}\Delta g)^{-1} < N/(2\pi) < (1 - (1 + e^{-2})\Delta g)^{-1}$ , cf. Eq. (5). However, simulations demonstrate that the fundamental modes trapped in the annulus are stable only in a narrow interval near the left edge of this region—for instance, at  $5.78 < N < 5.92$  for  $\rho = 0.2$ ,  $\Delta g = 0.5$ . At larger  $N$ , the modes are destabilized by perturbations which break the axial symmetry, driving the soliton's peak off the center and, eventually, causing the collapse. This instability (similar to that found for the annulus embedded into the linear medium [22]) is not comprised by the VK criterion, which pertains solely to axisymmetric perturbations.

We have also checked the dynamics of modes with embedded vorticity in this 2D setting, concluding that, as well as in other cases when the transverse or longitudinal modulation of the SF cubic nonlinearity is employed [22], [23], [29]–[31], vortex solitons cannot be stabilized against splitting by azimuthal perturbations (on the other hand, solitary vortices supported by the transverse modulation of the self-defocusing nonlinearity growing at  $r \rightarrow \infty$  can be easily made stable [25]).

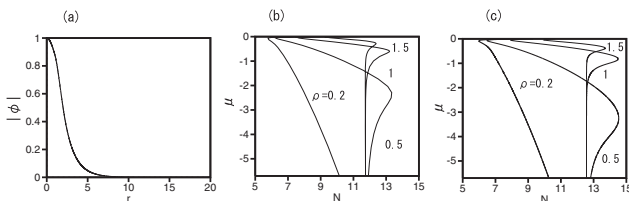


FIG. 2: (a) The profile of a fundamental soliton trapped in the annulus with  $\rho = 1$ ,  $\Delta g = 0.5$  and  $\mu = -0.216$ ,  $N = 9.82$ . (b) The propagation constant of numerically found families of the trapped modes vs.  $N$  in annuli with nonlinearity contrast  $\Delta g = 0.5$  and different inner radii  $\rho$ . (c) The same as in (b), but as produced by the VA.

The 1D counterpart of the critical collapse occurs in the NLS equation with the quintic term,  $i\phi_z = -(1/2)\phi_{xx} - |\phi|^4\phi$  [18, 33, 34]. In optics, this model can be realized experimentally in colloids [32]. The quintic equation gives rise to the 1D version of the Townes soliton,  $\phi = (-3\mu)^{1/4} e^{-i\mu z} \sqrt{\text{sech}(\sqrt{-8\mu}x)}$ , with the constant total power:  $N_0 \equiv \sqrt{3/8\pi}$  for any  $\mu < 0$ . The simplest 1D counterpart of the 2D setting considered above

is provided by the spatially modulated coefficient in front of the quintic term:  $g(x) = 1 + \Delta g \cdot \delta(x)$ , with  $\Delta g > 0$  [for a similar radial modulation of the cubic SF nonlinearity in the 2D model,  $g(r) = g_0\delta(r - r_0)$ , exact solutions for trapped modes can be found, but they are unstable against azimuthal perturbations [22]]. Then, the 1D Townes solitons are replaced by exact solutions  $\phi = (-3\mu)^{1/4} e^{-i\mu z} \sqrt{\text{sech}(\sqrt{-8\mu}(|x| + \xi))}$ , with  $\xi$  defined by relation  $\sinh(2\sqrt{-8\mu}\xi) = 3\Delta g\sqrt{-2\mu}$ , and total power  $N_{1D} = \sqrt{6} \tan^{-1} \left\{ \sqrt{\sqrt{-18(\Delta g)^2\mu + 1} + 3\Delta g\sqrt{-2\mu}} \right\}$  taking values  $\sqrt{3/8\pi} < N_{1D} < \sqrt{3/2\pi}$ . The entire family satisfies the VK stability condition,  $dN/d\mu < 0$ , and is indeed stable.

In conclusion, it is demonstrated that the axisymmetric modulation of the strength of the SF cubic nonlinearity with sharp edges stabilizes a family of 2D solitons against the critical collapse. For small contrast  $\Delta g$ , the width of the stability interval of the soliton's norm scales as  $\Delta N \sim \Delta g$ . For the annulus, the stability interval is strongly reduced by azimuthal perturbations.

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